

OPTIMAL CONVERGENCE RATES FOR THE THREE-DIMENSIONAL TURBULENT FLOW EQUATIONS

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ABSTRACT. In this paper we are concerned with the convergence rate of solutions to the three-dimensional turbulent flow equations. By combining the L^p - L^q estimates for the linearized equations and an elaborate energy method, the convergence rates are obtained in various norms for the solution to the equilibrium state in the whole space, when the initial perturbation of the equilibrium state is small in H^3 -framework. More precisely, the optimal convergence rates of the solutions and its first order derivatives in L^2 -norm are obtained when the L^p -norm of the perturbation is bounded for some $p \in [1, \frac{6}{5})$.

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1. INTRODUCTION

We consider in this work the turbulent flow equations for compressible flows on \mathbb{R}^3 ,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \Delta u - \nabla \operatorname{div} u + \nabla p = -\frac{2}{3} \nabla(\rho k), \\ (\rho h)_t + \operatorname{div}(\rho u h) - \Delta h = \frac{Dp}{Dt} + S_k, \\ (\rho k)_t + \operatorname{div}(\rho u k) - \Delta k = G - \rho \varepsilon, \\ (\rho \varepsilon)_t + \operatorname{div}(\rho u \varepsilon) - \Delta \varepsilon = \frac{C_1 G \varepsilon}{k} - \frac{C_2 \rho \varepsilon^2}{k}, \\ (\rho, u, h, k, \varepsilon)(x, t)|_{t=0} = (\rho_0(x), u_0(x), h_0(x), k_0(x), \varepsilon_0(x)), \end{cases} \quad (1.1)$$

with $S_k = [\mu(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}) - \frac{2}{3} \delta_{ij} \mu \frac{\partial u^k}{\partial x_k}] \frac{\partial u^i}{\partial x_j} + \frac{\mu_t}{\rho^2} \frac{\partial p}{\partial x_j} \frac{\partial \rho}{\partial x_j}$, $G = \frac{\partial u^i}{\partial x_j} [\mu_e (\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}) - \frac{2}{3} \delta_{ij} (\rho k + \mu_e \frac{\partial u^k}{\partial x_k})]$, where δ_{ij} is given by $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$, dynamic viscosity μ and the eddy viscosity μ_t are positive constants satisfying $\mu + \mu_t = \mu_e$, and C_1, C_2 are also two adjustable positive constants.

Here ρ , u , h , k and ε denote the density, velocity, total enthalpy, turbulent kinetic energy and rate of viscous dissipation, respectively. The pressure p is a smooth function of ρ . In this paper, without loss of generality, we have renormalized some constants to be 1. The system (1.1) is formed by combining effect of turbulence on time-averaged Navier-Stokes equations with the k - ε model equations.

All flows encountered in engineering practice, both simple ones such as two-dimensional jets, wakes, pipe flows and flat plate boundary layers and more complicated three-dimensional ones, become unstable above a certain Reynolds number. At low Reynolds numbers flows are laminar. Flows in the laminar regime are described by the continuity and Navier-Stokes equations which have been studied by

many people [1, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 19, 23, 20, 24, 25, 26, 27, 28]. At high Reynolds numbers flows are observed to become turbulent. A chaotic and random state of motion develops in which the velocity and pressure change continuously with time within substantial regions of flow. More precisely, at values of the Reynolds number above Re_{crit} a complicated series of events takes place which eventually leads to a radical change of the flow character. In the final state the flow behavior is random and chaotic. The motion becomes intrinsically unsteady even with constant imposed boundary conditions. The velocity and all other flow properties vary in a random and chaotic way. Turbulence stands out as a prototype of multi-scale phenomenon that occurs in nature. It involves wide ranges of spatial and temporal scales which makes it very difficult to study analytically and prohibitively expensive to simulate computationally. Many, if not most, flows of engineering significance are turbulent, so the turbulent flow regime is not just of theoretical interest. Up to now, although many physicists and mathematicians studied turbulent flows, there are not any general theory suitable for them. Fluid engineers need access to viable tools capable of representing the effects of turbulence.

This paper is devoted to study decay rates for the system (1.1) and proves the optimal convergence rates of its solutions under suitable assumptions. Bian-Guo [3] has obtained the global existence of smooth solutions to the system (1.1) under the condition that the initial data are close to the equilibrium state in H^3 -framework. More precisely, this result is expressed in the following.

Proposition 1.1. *Assume that initial data are close enough to the constant state $(\bar{\rho}, 0, 0, \bar{k}, 0)$, i.e. there exists a constant δ_0 such that if*

$$\|(\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0)\|_{H^3(\mathbb{R}^3)} \leq \delta_0, \quad (1.2)$$

then the system (1.1) admits a unique smooth solution $(\rho, u, h, k, \varepsilon)$ such that for any $t \in [0, \infty)$,

$$\begin{aligned} & \|(\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon)\|_{H^3}^2 + \int_0^t \|\nabla \rho\|_{H^2}^2 + \|(\nabla u, \nabla h, \nabla k, \nabla \varepsilon)\|_{H^3}^2 ds \\ & \leq C \|(\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0)\|_{H^3}^2, \end{aligned} \quad (1.3)$$

where C is a positive constant.

Based on this stability result, the main purpose in this paper is to investigate the optimal convergence rates in time to the stationary solution. We remark that the convergence rate is an important topic in the study of the fluid dynamics for the purpose of the computation [13, 18]. The main idea in this paper is to combine the L^p - L^q estimates for the linearized equations and an improved energy method which includes the estimation on the higher power of L^2 -norm of solutions. By doing this, the optimal convergence rates for the solutions to the nonlinear problem (1.1) in various norms can be obtained and are stated in the following theorem.

Theorem 1.2. *Let δ_0 be the constant defined in Proposition 1.1. There exist constants $\delta_1 \in (0, \delta_0)$ and $C > 0$ such that the following holds. For any $\delta \leq \delta_1$, if*

$$\|(\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0)\|_{H^3(\mathbb{R}^3)} \leq \delta, \quad (1.4)$$

and for some $p \in [1, \frac{6}{5})$,

$$\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0 \in L^p(\mathbb{R}^3), \quad (1.5)$$

then the smooth solution $(\rho, u, h, k, \varepsilon)$ in Proposition 1.1 enjoys the estimates for $t \in [0, \infty)$,

$$\|(\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon)\|_q \leq C(1+t)^{-\sigma(p,q;0)}, \quad 2 \leq q \leq 6, \quad (1.6)$$

$$\|(\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon)\|_\infty \leq C(1+t)^{-\sigma(p,2;1)}, \quad (1.7)$$

$$\|\nabla(\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon)\|_{H^2} \leq C(1+t)^{-\sigma(p,2;1)}, \quad (1.8)$$

$$\|(\rho_t, u_t, h_t, k_t, \varepsilon_t)\|_2 \leq C(1+t)^{-\sigma(p,2;1)}, \quad (1.9)$$

where $\sigma(p, q; l)$ are defined by

$$\sigma(p, q; l) = \frac{3}{2} \left(\frac{1}{p} - \frac{1}{q} \right) + \frac{l}{2}. \quad (1.10)$$

Remark 1.3. (1.5) shows that the perturbation of initial data around the constant state $(\bar{\rho}, 0, 0, \bar{k}, 0)$ is bounded in L^p -norm, for some $p \in [1, \frac{6}{5})$, which need not be small.

Remark 1.4. The linearized equations of (1.1) around the constant state $(\bar{\rho}, 0, 0, \bar{k}, 0)$ take the following form:

$$\begin{cases} a_t + \gamma \operatorname{div} v = 0, \\ v_t + \gamma \nabla a - \lambda \Delta v - \lambda \nabla \operatorname{div} v = 0, \\ h_t - \lambda \Delta h = 0, \\ m_t - \lambda \Delta m = 0, \\ \varepsilon_t - \lambda \Delta \varepsilon = 0, \end{cases} \quad (1.11)$$

where γ, λ are positive constants which will be given precisely in Section 2. Compared to the decay estimates of the solutions to the above linearized equations by using Fourier analysis [22] stated in Lemma 2.1 in the next section, Theorem 1.2 gives the optimal decay rates for the solution in L^q -norm, for any $2 \leq q \leq 6$, and its first order estimates in L^2 -norm. Note that the convergence rates of the derivatives of higher order in L^2 -norm and the solution in L^∞ -norm are not the same as those for linearized equations.

Remark 1.5. We mainly use the method in [12] to prove Theorem 1.2. But our problem is much more difficult because of the strong coupling between velocity, total enthalpy, turbulent kinetic energy and rate of viscous dissipation. Moreover, our result is better than that in [12]. Here we assume L^p -norm of initial data $\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0$ is bounded, for some $p \in [1, \frac{6}{5})$, instead of $\|(\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0)\|_{L^1} < \infty$.

Notation: Throughout the paper, C stands for a general constant, and may change from line to line. The norm $\|(A, B)\|_X$ is equivalent to $\|A\|_X + \|B\|_X$ and $\|A\|_{X \cap Y} = \|A\|_X + \|A\|_Y$. The norms in the Sobolev Spaces $H^m(\mathbb{R}^3)$ and $W^{m,q}(\mathbb{R}^3)$ are denoted respectively by $\|\cdot\|_{H^m}$ and $\|\cdot\|_{W^{m,q}}$ for $m \geq 0, q \geq 1$. In particular, for $m = 0$, we will simply use $\|\cdot\|_p$. Moreover, $\langle \cdot, \cdot \rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$. Finally,

$$\nabla = (\partial_1, \partial_2, \partial_3), \quad \partial_i = \partial_{x_i}, \quad i = 1, 2, 3,$$

and for any integer $l \geq 0$, $\nabla^l f$ denotes all derivatives up to l -order of the function f . And for multi-indices α, β and ξ

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \beta = (\beta_1, \beta_2, \beta_3), \quad \xi = (\xi_1, \xi_2, \xi_3),$$

we use

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad |\alpha| = \sum_{i=1}^3 \alpha_i,$$

and $C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!}$ when $\beta \leq \alpha$.

2. PRELIMINARIES

We will reformulate the problem (1.1) as follows. Set

$$\gamma = \sqrt{p'(\bar{\rho}) + \bar{k}}, \quad \lambda = \frac{1}{\bar{\rho}}.$$

Introducing new variables by

$$a = \rho - \bar{\rho}, \quad v = \frac{1}{\gamma\lambda}, \quad h = h, \quad m = k - \bar{k}, \quad \varepsilon = \varepsilon,$$

the initial value problem (1.1) is reformulated as

$$\begin{cases} a_t + \gamma \operatorname{div} v = F_1, \\ v_t + \gamma \nabla a - \lambda \Delta v - \lambda \nabla \operatorname{div} v = F_2, \\ h_t - \lambda \Delta h = F_3, \\ m_t - \lambda \Delta m = F_4, \\ \varepsilon_t - \lambda \Delta \varepsilon = F_5, \\ (a, v, h, m, \varepsilon)(x, 0) = (a_0, v_0, h_0, m_0, \varepsilon_0) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{cases} \quad (2.1)$$

where

$$\begin{aligned} F_1 &= -\gamma \lambda \operatorname{div}(au), \\ F_2 &= \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)(\Delta v + \nabla \operatorname{div} v) - \frac{1}{\gamma\lambda} \left(\frac{p'(a + \bar{\rho})}{a + \bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m + \bar{k})}{3(a + \bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}} \right) \nabla a \\ &\quad - \frac{2}{3\gamma\lambda} \nabla m, \\ F_3 &= \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right) \Delta h - \gamma \lambda p'(a + \bar{\rho}) \operatorname{div} v + \frac{1}{a + \bar{\rho}} S_k^1 - \gamma \lambda v \cdot \nabla h, \\ F_4 &= \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right) \Delta m + \frac{1}{a + \bar{\rho}} G^1 - \varepsilon - \gamma \lambda v \cdot \nabla m, \\ F_5 &= \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right) \Delta \varepsilon + \frac{C_1 G^1 \varepsilon}{(a + \bar{\rho})(m + \bar{k})} - \frac{C_2 \varepsilon^2}{m + \bar{k}} S_k - \gamma \lambda v \cdot \nabla \varepsilon, \end{aligned} \quad (2.2)$$

with S_k^1 and G^1 the corresponding S_k and G in the variables of $(a, v, h, m, \varepsilon)$.

Let $U = (a, v)$, $U_0 = (a_0, v_0)$, $F = (F_1, F_2)$,

$$A = \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma \nabla & \lambda \Delta + \lambda \nabla \operatorname{div} \end{pmatrix}$$

and $E(t)$ be the semigroup generated by the linear operator A , then we can rewrite the solution for the first two equations of the nonlinear problem (1.1) as

$$U(t) = E(t)U_0 + \int_0^t E(t-s)F ds. \quad (2.3)$$

The semigroup $E(t)$ has the following properties on the decay in time, which can be found in [21, 22] and will be applied to the integral formula (2.3).

Lemma 2.1. *Let $l \geq 0$ be an integer and $1 \leq p \leq 2 \leq q < \infty$. Then for any $t \geq 0$, it holds that*

$$\|\nabla^l E(t)U_0\|_q \leq C(1+t)^{-\sigma(p,q;l)}\|U_0\|_{L^p \cap H^l}, \quad (2.4)$$

with $\sigma(p, q; l)$ defined by (1.10).

To treat the last three equations, we introduce the semigroup $S(t)$ generated by $\lambda\Delta$, then (2.1)₃-(2.1)₅ become

$$\begin{aligned} h(t) &= S(t)h_0 + \int_0^t S(t-s)F_3 ds, \\ m(t) &= S(t)m_0 + \int_0^t S(t-s)F_4 ds, \\ \varepsilon(t) &= S(t)\varepsilon_0 + \int_0^t S(t-s)F_5 ds. \end{aligned} \quad (2.5)$$

We state the large-time behavior of solutions to the last three equations of the system (2.1) as the following lemma which can be obtained by direct calculation or can refer to [29].

Lemma 2.2. *For the solution (h, m, ε) of the last three equations of the system (2.1) with Cauchy data $h(x, 0) = h_0$, $m(x, 0) = m_0$, $\varepsilon(x, 0) = \varepsilon_0$, there exists a constant C such that*

$$\begin{aligned} \|(\nabla^l h, \nabla^l m, \nabla^l \varepsilon)\|_q &\leq C(1+t)^{-\sigma(p,q;l)}\|(h_0, m_0, \varepsilon_0)\|_p \\ &+ C \int_0^t (1+t-s)^{-\sigma(p,q;l)}\|(F_3, F_4, F_5)\|_p, \quad l = 0, 1, \end{aligned} \quad (2.6)$$

for any $t \geq 0$, $1 \leq p, q \leq +\infty$, as well as σ is defined by (1.10).

For later use we list some Sobolev inequalities as follows, cf. [2, 11].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^3$ be the whole space \mathbb{R}^3 , or half space \mathbb{R}_+ or the exterior domain of a bounded region with smooth boundary. Then*

- (i) $\|f\|_{L^6(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$, for $f \in H^1(\Omega)$.
- (ii) $\|f\|_{L^p(\Omega)} \leq C\|f\|_{H^1(\Omega)}$, for $2 \leq p \leq 6$.
- (iii) $\|f\|_{C^0(\bar{\Omega})} \leq C\|f\|_{W^{1,p}(\Omega)} \leq C\|\nabla f\|_{H^1(\Omega)}$, for $f \in H^2(\Omega)$.
- (iv) $|\int_{\Omega} f \cdot g \cdot h dx| \leq \varepsilon\|\nabla f\|_{L^2}^2 + \frac{C}{\varepsilon}\|g\|_{H^1}^2\|h\|_{L^2}^2$, for $\varepsilon > 0$, $f, g \in H^1(\Omega)$, $h \in L^2(\Omega)$.
- (v) $|\int_{\Omega} f \cdot g \cdot h dx| \leq \varepsilon\|g\|_{L^2}^2 + \frac{C}{\varepsilon}\|\nabla f\|_{H^1}^2\|h\|_{L^2}^2$, for $\varepsilon > 0$, $f \in H^2(\Omega)$, $g, h \in L^2(\Omega)$.

Finally, the following elementary inequality [12] will also be used.

Lemma 2.4. *If $r_1 > 1$ and $r_2 \in [0, r_1]$, then it holds that*

$$\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_2} ds \leq C_1(r_1, r_2)(1+t)^{-r_2}, \quad (2.7)$$

where $C_1(r_1, r_2)$ is defined by

$$C_1(r_1, r_2) = \frac{2^{r_2+1}}{r_1 - 1}. \quad (2.8)$$

3. BASIC ESTIMATES

In this section we shall establish two basic inequalities for the proof of the optimal convergence rates in section 4. One inequality is the decay rate of the first order derivatives, while the other is the energy estimate.

Lemma 3.1. *Let $W = (U, h, m, \varepsilon)$ be the solution to the problem (2.1), then under the assumptions of Theorem 1.2, we have*

$$\|\nabla W\|_2 \leq CE_0(1+t)^{-\sigma(p,2;1)} + C \int_0^t (1+t-s)^{-\sigma(p,2;1)} \|\nabla W\|_{H^2} ds, \quad (3.1)$$

with $E_0 = \|W(0)\|_{L^p \cap H^1} = \|(U_0, h_0, m_0, \varepsilon_0)\|_{L^p \cap H^1} = \|(a_0, v_0, h_0, m_0, \varepsilon_0)\|_{L^p \cap H^1}$ and $1 \leq p < \frac{6}{5}$.

Proof. Let $l = 1$ in (2.4), we then have from (2.3)

$$\|\nabla U(t)\|_2 \leq CE_0(1+t)^{-\sigma(p,2;1)} + C \int_0^t (1+t-s)^{-\sigma(p,2;1)} \|(F_1, F_2)\|_{L^p \cap H^1} ds,$$

which together with (2.6) implies that

$$\begin{aligned} \|\nabla W(t)\|_2 &\leq CE_0(1+t)^{-\sigma(p,2;1)} + C \int_0^t (1+t-s)^{-\sigma(p,2;1)} [\|(F_1, F_2)\|_{L^p \cap H^1} \\ &\quad + \|(F_3, F_4, F_5)\|_p] ds. \end{aligned} \quad (3.2)$$

For $\|(F_1, F_2)\|_{L^p \cap H^1}$, we estimate as follows. For $1 \leq p < \frac{6}{5}$, the term in F_1 can be estimated as

$$\begin{aligned} \|\gamma \lambda \operatorname{div}(av)\|_p &\leq C(\|\partial_i a v^i\|_p + \|a \partial_i v^i\|_p) \\ &\leq C\|v\|_{\frac{2p}{2-p}} \|\nabla a\|_2 + C\|\nabla v\|_2 \|a\|_{\frac{2p}{2-p}} \\ &\leq C\delta \|(\nabla a, \nabla v)\|_2. \end{aligned}$$

With the help of Hölder inequality and Lemma 2.3, it holds that

$$\begin{aligned} \|\gamma \lambda \operatorname{div}(av)\|_{H^1} &\leq C(\|\partial_i a v^i\|_2 + \|a \partial_i v^i\|_2 + \|\nabla \operatorname{div}(av)\|_2) \\ &\leq C\delta \|(\nabla a, \nabla v)\|_{H^1}. \end{aligned}$$

Similarly, the terms in F_2 can be estimated as

$$\begin{aligned} \|(\frac{1}{\rho} - \frac{1}{\bar{\rho}})(\Delta v + \nabla \operatorname{div} v)\|_p &\leq C\|\nabla^2 v\|_2 \|a\|_{\frac{2p}{2-p}} \leq C\delta \|\nabla^2 v\|_2, \\ \|(\frac{1}{\rho} - \frac{1}{\bar{\rho}})(\Delta v + \nabla \operatorname{div} v)\|_{H^1} &\leq C\|(\frac{1}{\rho} - \frac{1}{\bar{\rho}})(\Delta v + \nabla \operatorname{div} v)\|_2 + \|\nabla((\frac{1}{\rho} - \frac{1}{\bar{\rho}})(\Delta v + \nabla \operatorname{div} v))\|_2 \\ &\leq C(\|a\|_\infty \|\nabla^2 v\|_2 + \|a\|_\infty \|\nabla^3 v\|_2 + \|\nabla a\|_\infty \|\nabla^2 v\|_2) \\ &\leq C\delta \|\nabla v\|_{H^2}, \end{aligned}$$

$$\begin{aligned} \|\gamma \lambda (\frac{p'(a+\bar{\rho})}{a+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m+\bar{k})}{3(a+\bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}}) \nabla a\|_p &\leq C(\|\nabla a\|_2 \|a\|_{\frac{2p}{2-p}} + \|\nabla a\|_2 \|m\|_{\frac{2p}{2-p}}) \\ &\leq C\delta \|\nabla a\|_2, \end{aligned}$$

$$\begin{aligned}
& \left\| -\frac{1}{\gamma\lambda} \left(\frac{p'(a+\bar{\rho})}{a+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m+\bar{k})}{3(a+\bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}} \right) \nabla a \right\|_{H^1} \\
& \leq C \left(\left\| \left(\frac{p'(a+\bar{\rho})}{a+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m+\bar{k})}{3(a+\bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}} \right) \nabla a \right\|_2 \right. \\
& \quad \left. + \left\| \nabla \left[\left(\frac{p'(a+\bar{\rho})}{a+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m+\bar{k})}{3(a+\bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}} \right) \nabla a \right] \right\|_2 \right) \\
& \leq C (\|a\nabla a + m\nabla a\|_2 + \|a\nabla^2 a + m\nabla^2 a + (\nabla a)^2 + \nabla a \nabla m\|_2) \\
& \leq C\delta \|\nabla a\|_{H^2},
\end{aligned}$$

$$\left\| -\frac{2}{3\gamma\lambda} \nabla m \right\|_p \leq \|\nabla m\|_2 \|m + \bar{k}\|_6 \left\| \frac{1}{m + \bar{k}} \right\|_{\frac{3p}{3-2p}} \leq C\delta \|\nabla m\|_2,$$

$$\begin{aligned}
& \left\| -\frac{2}{3\gamma\lambda} \nabla m \right\|_{H^1} \leq C (\|\nabla m\|_2 + \|\nabla^2 m\|_2) \\
& \leq C (\|\nabla m\|_3 \|m + \bar{k}\|_6 \left\| \frac{1}{m + \bar{k}} \right\|_\infty + \|\nabla^2 m\|_3 \|m + \bar{k}\|_6 \left\| \frac{1}{m + \bar{k}} \right\|_\infty) \\
& \leq C\delta \|\nabla m\|_{H^2}.
\end{aligned}$$

Thus we have

$$\|(F_1, F_2)\|_{L^p \cap H^1} \leq C\delta \|(\nabla a, \nabla v, \nabla m)\|_{H^2}. \quad (3.3)$$

Next, we estimate for $\|(F_3, F_4, F_5)\|_p$. Almost as same as the estimation of $\|(F_1, F_2)\|_{L^p \cap H^1}$, we get

$$\|(F_3, F_4, F_5)\|_p \leq C\delta \|(\nabla a, \nabla v, \nabla h, \nabla m, \nabla \varepsilon)\|_{H^2}. \quad (3.4)$$

Inserting (3.3) and (3.4) into (3.2), we complete the proof of Lemma 3.1. \square

Lemma 3.2. *Let $W = (U, h, m, \varepsilon)$ be the solution to the problem (2.1), then under the assumptions of Theorem 1.2, if $\delta > 0$ is sufficiently small then it holds that*

$$\frac{dM(t)}{dt} + \|\nabla^2 a\|_{H^1}^2 + \|\nabla^2(v, h, m, \varepsilon)\|_{H^2}^2 \leq C\delta \|\nabla(a, v, h, m, \varepsilon)\|_2^2, \quad (3.5)$$

where $M(t)$ is equivalent to $\|\nabla(a, v, h, m, \varepsilon)\|_{H^2}^2$, i.e., there exists a positive constant C_2 such that

$$C_2^{-1} \|\nabla(a, v, h, m, \varepsilon)\|_{H^2}^2 \leq M(t) \leq C_2 \|\nabla(a, v, h, m, \varepsilon)\|_{H^2}^2. \quad (3.6)$$

Proof. Let α be any multi-index with $1 \leq |\alpha| \leq 3$. Applying the operator ∂_x^α to (2.1) and then taking inner product with $\partial_x^\alpha a$, $\partial_x^\alpha v$, $\partial_x^\alpha h$, $\partial_x^\alpha m$ and $\partial_x^\alpha \varepsilon$, one gets

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha(a, v, h, m, \varepsilon)\|_2^2 + \lambda \|\nabla \partial_x^\alpha v\|_2^2 + \lambda \|\operatorname{div} \partial_x^\alpha v\|_2^2 + \lambda \|\nabla \partial_x^\alpha(h, m, \varepsilon)\|_2^2 \\
& = \langle \partial_x^\alpha a, \partial_x^\alpha F_1 \rangle + \langle \partial_x^\alpha v, \partial_x^\alpha F_2 \rangle + \langle \partial_x^\alpha h, \partial_x^\alpha F_3 \rangle + \langle \partial_x^\alpha m, \partial_x^\alpha F_4 \rangle \\
& \quad + \langle \partial_x^\alpha \varepsilon, \partial_x^\alpha F_5 \rangle \\
& =: J_1^\alpha(t) + J_2^\alpha(t) + J_3^\alpha(t) + J_4^\alpha(t) + J_5^\alpha(t),
\end{aligned}$$

where $J_i^\alpha(t)$, $i = 1, 2, 3, 4, 5$, are the corresponding terms in the above equation which will be estimated as follows.

Now let's estimate for $J_1^\alpha(t)$. It follows from Lemma 2.3 that

$$\begin{aligned} J_1^1 &\sim \langle \partial_j \partial_i (av^i), \partial_j a \rangle \sim \langle \partial_j \partial_i av^i + \partial_i a \partial_j v^i + \partial_j a \partial_i v^i + \partial_i \partial_j v^i a, \partial_j a \rangle \\ &\leq C \|\nabla^2 a\|_2 \|\nabla a\|_2 \|v\|_\infty + \|\nabla a\|_2^2 \|\nabla v\|_\infty + \|\nabla^2 v\|_2 \|\nabla a\|_2 \|a\|_\infty \\ &\leq \delta \|\nabla a\|_2^2 + \frac{C}{\delta} \|\nabla^2 a\|_2^2 \|v\|_{H^2}^2 + \frac{C}{\delta} \|\nabla^2 v\|_2^2 \|a\|_{H^2}^2, \end{aligned}$$

$$\begin{aligned} J_1^2 &\sim \langle \partial_x^\beta \partial_i (av^i), \partial_x^{\alpha+\xi} a \rangle \sim \langle \nabla^2 av + \nabla a \nabla v + \nabla^2 va, \nabla^3 a \rangle \\ &\leq \delta \|\nabla^3 a\|_2^2 + \frac{C}{\delta} (\|\nabla^2 a\|_2^2 \|v\|_\infty^2 + \|\nabla a\|_2^2 \|\nabla v\|_\infty^2 + \|\nabla^2 v\|_2^2 \|a\|_\infty^2) \\ &\leq \delta \|\nabla^3 a\|_2^2 + \frac{C}{\delta} (\|\nabla^2 a\|_2^2 \|v\|_{H^2}^2 + \|\nabla a\|_2^2 \|\nabla v\|_{H^2}^2 + \|\nabla^2 v\|_2^2 \|a\|_{H^2}^2), \end{aligned}$$

where $\alpha = \beta + \xi$, with $|\beta| = |\xi| = 1$.

For J_1^3 , we have from multi-indices β and ξ with $|\beta| = 2$, $|\xi| = 1$,

$$\begin{aligned} J_1^3 &\sim \langle \partial_x^{\beta+\xi} \partial_i (av^i), \partial_x^{\beta+\xi} a \rangle \\ &\sim - \int_{\mathbb{R}^3} (\partial_x^{\beta+\xi} a)^2 \operatorname{div} v dx + \langle \nabla a \nabla^3 v + \nabla^3 a \nabla v + \nabla^2 a \nabla^2 v + a \nabla^4 v, \nabla^3 a \rangle \\ &\leq \delta \|\nabla^3 a\|_2^2 + \frac{C}{\delta} (\|\nabla^2 a\|_3^2 \|\nabla^2 v\|_6^2 + \|\nabla^3 v\|_2^2 \|\nabla a\|_\infty^2 + \|\nabla^4 v\|_2^2 \|a\|_\infty^2) \\ &\leq \delta \|\nabla^3 a\|_2^2 + \frac{C}{\delta} (\|\nabla^2 a\|_2^2 \|v\|_{H^2}^2 + \|\nabla a\|_2^2 \|\nabla v\|_{H^2}^2 + \|\nabla^2 v\|_2^2 \|a\|_{H^2}^2). \end{aligned}$$

Hence

$$J_1^\alpha \leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha v\|_2^2. \quad (3.7)$$

Similarly, by Lemma 2.3 and Hölder inequality, J_2^α can be estimated as

$$\begin{aligned} J_2^1 &\sim \langle \partial_j [(\frac{1}{\rho} - \frac{1}{\bar{\rho}})(\Delta v + \nabla \operatorname{div} v) - \frac{1}{\gamma\lambda}(\frac{p'(a+\bar{\rho})}{a+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m+\bar{k})}{3(a+\bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}})\nabla a \\ &\quad - \frac{2}{3\gamma\lambda}\nabla m], \partial_j v \rangle \\ &\sim \langle \partial_j a(\Delta v + \nabla \operatorname{div} v) + a(\partial_j \Delta v + \partial_j \nabla \operatorname{div} v) + \partial_j a \cdot \nabla a + \partial_j m \cdot \nabla a + a \partial_j \nabla a \\ &\quad + m \partial_j \nabla a + \partial_j \nabla m, \partial_j v \rangle \\ &\leq C(\|\nabla a\|_3 \|\nabla v\|_6 \|\nabla^2 v\|_2 + \|\nabla^3 v\|_2 \|\nabla v\|_2 \|a\|_\infty + \|\nabla(a, m)\|_3 \|\nabla a\|_6 \|\nabla v\|_2 \\ &\quad + \|\nabla^2 a\|_2 \|\nabla v\|_2 \|m\|_\infty + \|\nabla^2 a\|_2 \|\nabla v\|_2 \|a\|_\infty + \|\nabla^2 m\|_2 \|\nabla v\|_3 \|\nabla m\|_2) \\ &\leq \delta(\|\nabla^2 v\|_2^2 + \|\nabla^3 v\|_2^2 + \|\nabla^2(a, m)\|_2^2) + \frac{C}{\delta} (\|\nabla v\|_2^2 \|(a, m)\|_{H^2}^2 + \|\nabla v\|_{H^1}^2 \|\nabla m\|_2^2) \\ &\leq C\delta \|\nabla^2(a, m)\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha v\|_2^2, \end{aligned}$$

$$\begin{aligned}
J_2^2 &\sim \langle \partial_x^\beta [(\frac{1}{\rho} - \frac{1}{\bar{\rho}})(\Delta v + \nabla \operatorname{div} v) - \frac{1}{\gamma\lambda}(\frac{p'(a+\bar{\rho})}{a+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m+\bar{k})}{3(a+\bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}})\nabla a \\
&\quad - \frac{2}{3\gamma\lambda}\nabla m], \partial_x^{\alpha+\xi} v \rangle \\
&\sim \langle \nabla a(\Delta v + \nabla \operatorname{div} v) + a\nabla^3 v + \nabla a\nabla a + \nabla a\nabla m + a\nabla^2 a + m\nabla^2 a + \nabla^2 m, \nabla^3 v \rangle \\
&\leq C(\|\nabla a\|_3\|\nabla^2 v\|_6\|\nabla^3 v\|_2 + \|\nabla^3 v\|_2^2\|a\|_\infty + \|\nabla a\|_3\|\nabla a\|_6\|\nabla^3 v\|_2 \\
&\quad + \|\nabla a\|_6\|\nabla^3 v\|_2\|\nabla m\|_3 + \|\nabla^2 a\|_2\|\nabla^3 v\|_2\|(a, m)\|_\infty + \|\nabla^2 m\|_3\|\nabla^2 v\|_2\|m+\bar{k}\|_6) \\
&\leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha(v, m)\|_2^2, \text{ with } \alpha = \beta + \xi, \ |\beta| = |\xi| = 1,
\end{aligned}$$

$$\begin{aligned}
J_2^3 &\sim \langle \partial_x^\beta [(\frac{1}{\rho} - \frac{1}{\bar{\rho}})(\Delta v + \nabla \operatorname{div} v) - \frac{1}{\gamma\lambda}(\frac{p'(a+\bar{\rho})}{a+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m+\bar{k})}{3(a+\bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}})\nabla a \\
&\quad - \frac{2}{3\gamma\lambda}\nabla m], \partial_x^{\alpha+\xi} v \rangle \\
&\sim \langle a\nabla^4 v + \nabla a\nabla^3 v + \nabla^2 a\nabla^2 v + a\nabla^3 a + \nabla a\nabla^2 a + \nabla m\nabla^2 a + \nabla^2 m\nabla a \\
&\quad + m\nabla^3 a + \nabla^3 m, \nabla^4 v \rangle \\
&\leq C\delta\|\nabla^4 v\|_2^2 + \frac{C}{\delta}(\|\nabla a\|_\infty^2\|\nabla^3 v\|_2^2 + \|\nabla^2 v\|_3^2\|\nabla^2 a\|_6^2 + \|\nabla^3 a\|_2^2\|a\|_\infty^2 \\
&\quad + \|\nabla^2 a\|_2^2\|\nabla(a, m)\|_\infty^2 + \|\nabla^2 m\|_2^2\|\nabla a\|_\infty^2 + \|\nabla^3 a\|_2^2\|m\|_\infty^2 + \|\nabla^3 m\|_3\|\nabla m\|_2 \\
&\leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha(v, m)\|_2^2, \text{ with } \alpha = \beta + \xi, \ |\beta| = 2, \ |\xi| = 1.
\end{aligned}$$

Incorporating the above estimates, it holds that

$$J_2^\alpha(t) \leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha(v, m)\|_2^2. \quad (3.8)$$

Moreover, $J_3^\alpha(t)$, $J_4^\alpha(t)$ and $J_5^\alpha(t)$ can be estimated similarly by using Lemma 2.3 and Hölder inequality,

$$|J_3^\alpha(t), J_4^\alpha(t), J_5^\alpha(t)| \leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha(v, h, m, \varepsilon)\|_2^2. \quad (3.9)$$

On the other hand, we apply ∂_x^α to (2.1)₂ with $1 \leq |\alpha| \leq 2$, and then take inner product with $\nabla \partial_x^\alpha a$ to yield

$$\langle \partial_x^\alpha v_t, \nabla \partial_x^\alpha a \rangle + \gamma \|\nabla \partial_x^\alpha a\|_2^2 = \lambda \langle \partial_x^\alpha \Delta v + \partial_x^\alpha \nabla \operatorname{div} v, \partial_x^\alpha \nabla a \rangle + \langle \partial_x^\alpha F_2, \partial_x^\alpha \nabla a \rangle, \quad (3.10)$$

and similarly from (2.1)₁,

$$\langle \partial_x^\alpha v, \nabla \partial_x^\alpha a_t \rangle = -\gamma \langle \partial_x^\alpha v, \nabla \partial_x^\alpha \operatorname{div} v \rangle + \langle \partial_x^\alpha v, \nabla \partial_x^\alpha F_1 \rangle. \quad (3.11)$$

Adding (3.10) and (3.11) together implies

$$\begin{aligned}
\frac{d}{dt} \langle \partial_x^\alpha v, \nabla \partial_x^\alpha a \rangle + \gamma \|\nabla \partial_x^\alpha a\|_2^2 &= \lambda \langle \partial_x^\alpha \Delta v + \partial_x^\alpha \nabla \operatorname{div} v, \partial_x^\alpha \nabla a \rangle \\
&\quad + \langle \partial_x^\alpha F_2, \partial_x^\alpha \nabla a \rangle - \gamma \langle \partial_x^\alpha v, \nabla \partial_x^\alpha \operatorname{div} v \rangle + \langle \partial_x^\alpha v, \nabla \partial_x^\alpha F_1 \rangle.
\end{aligned} \quad (3.12)$$

By Hölder inequality and similar to the estimation of $J_2^\alpha(t)$, the right hand side can be bounded by

$$\frac{\gamma}{2} \|\nabla \partial_x^\alpha a\|_2^2 + C \sum_{1 \leq |\alpha| \leq 2} \|\partial_x^\alpha \nabla v\|_{H^1}^2 + C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha (v, h, m, \varepsilon)\|_2^2. \quad (3.13)$$

Therefore, if we define

$$M(t) = C_1 \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha (a, v, h, m, \varepsilon)\|_2^2 + \sum_{1 \leq |\alpha| \leq 2} \langle \partial_x^\alpha v, \nabla \partial_x^\alpha a \rangle,$$

and choosing δ sufficiently small, then (3.7)-(3.9) and (3.13) imply that

$$\frac{dM(t)}{dt} + C_1 (\|\nabla^2 a\|_{H^1}^2 + \|\nabla^2 (v, h, m, \varepsilon)\|_{H^2}^2) \leq C\delta \|\nabla (a, v, h, m, \varepsilon)\|_2^2,$$

where C_1 is a positive constant independent of δ . Thus we arrive at the proof of the lemma. \square

4. OPTIMAL CONVERGENCE RATES

The optimal convergence rates can be proved by first improving the estimates given in Lemma 3.1 and Lemma 3.2 to the estimates on the L^2 -norms of solutions to higher power and then letting the power tend to infinity. By the inequalities (3.1) and (3.5), we have the following lemmas.

Lemma 4.1. *Let $W = (U, h, m, \varepsilon)$ be the solution to the problem (2.1), then under the assumptions of Theorem 1.2, if $\delta > 0$ is sufficiently small, then for any integer $n \geq 1$, and for some $p \in [1, \frac{6}{5})$, it holds that*

$$\int_0^t (1+s)^l \|\nabla W(s)\|_2^{2n} ds \leq (CE_0)^{2n} + (C\delta)^{2n} \int_0^t (1+s)^l \|\nabla^2 W(s)\|_{H^1}^{2n} ds, \quad (4.1)$$

where $l = 0, 1, \dots, N = [2n(\frac{3}{2p} - \frac{1}{4}) - 2]$, the constant E_0 is given in Lemma 3.1.

Lemma 4.2. *Let $W = (U, h, m, \varepsilon)$ be the solution to the problem (2.1), then under the assumptions of Theorem 1.2, if $\delta > 0$ is sufficiently small, then for any integer $n \geq 1$, and for some $p \in [1, \frac{6}{5})$, it holds that*

$$\begin{aligned} & (1+t)^l M(t)^n + n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ & \leq 2M(0)^n + (C_3 E_0)^{2n} + 8C_2 n \left(\frac{3}{2p} - \frac{1}{4}\right) \int_0^t (1+s)^{l-1} M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds, \end{aligned} \quad (4.2)$$

where $l = 0, 1, \dots, N = [2n(\frac{3}{2p} - \frac{1}{4}) - 2]$, the constant C_2 is given in Lemma 3.2, C_3 is independent of δ .

Remark 4.3. Lemma 4.1 and Lemma 4.2 are similar to that in [12]. But Lemma 4.1 and Lemma 4.2 in this work hold for general p with $p \in [1, \frac{6}{5})$. Note that lemmas hold only for $p = 1$ in [12]. For completeness, we state the proofs of Lemma 4.1 and Lemma 4.2 as follows.

Proof of Lemma 4.1. Fix any integer $n \geq 1$. By taking (3.1) to power $2n$ and multiplying it by $(1+t)^l$, $l = 0, 1, \dots, N$, integrating the resulting inequality over $[0, t]$ gives that

$$\begin{aligned} \int_0^t (1+\tau)^l \|\nabla W(\tau)\|_2^{2n} d\tau &\leq (CE_0)^{2n} \int_0^t (1+\tau)^{-2n(\frac{3}{2p}-\frac{1}{4})+l} d\tau \\ &+ (C\delta)^{2n} \int_0^t (1+\tau)^l \left[\int_0^\tau (1+\tau-s)^{-(\frac{3}{2p}-\frac{1}{4})} \|\nabla W(s)\|_{H^2} ds \right]^{2n} d\tau. \end{aligned} \quad (4.3)$$

It follows from the Hölder inequality that

$$\begin{aligned} &\left[\int_0^\tau (1+\tau-s)^{-(\frac{3}{2p}-\frac{1}{4})} \|\nabla W(s)\|_{H^2} ds \right]^{2n} \\ &\leq \left[\int_0^\tau (1+\tau-s)^{-r_1} (1+s)^{-r_2} ds \right]^{2n-1} \\ &\times \int_0^\tau (1+\tau-s)^{-\frac{4}{3}} (1+s)^l \|\nabla W(s)\|_{H^2}^{2n} ds, \end{aligned} \quad (4.4)$$

where

$$r_1 = \left(\frac{3}{2p} - \frac{1}{4} - \frac{2}{3n} \right) \frac{2n}{2n-1}$$

and

$$r_2 = \frac{l}{2n-1}.$$

Notice that $\frac{3}{p} - \frac{11}{6} \leq r_1 \leq \frac{3}{2p} - \frac{1}{4}$ and $r_2 \in [0, r_1]$ for $n \geq 1$ and $0 \leq l \leq N = [2n(\frac{3}{2p} - \frac{1}{4}) - 2]$, from Lemma 2.4, one deduces that

$$\int_0^\tau (1+\tau-s)^{-r_1} (1+s)^{-r_2} ds \leq C_1(r_1, r_2) (1+\tau)^{-r_2} \leq C(1+\tau)^{-r_2}, \quad (4.5)$$

where $C_1(r_1, r_2)$ given by (2.8) is bounded uniformly for $n \geq 1$. Hence, (4.3) together with (4.4) and (4.5) leads to

$$\begin{aligned} \int_0^t (1+\tau)^l \|\nabla W(\tau)\|_2^{2n} d\tau &\leq (CE_0)^{2n} \frac{1}{2n(\frac{3}{2p}-\frac{1}{4})-l-1} \\ &+ (C\delta)^{2n} \int_0^t (1+s)^l \|\nabla W(s)\|_{H^2}^{2n} \int_s^t (1+\tau-s)^{-\frac{4}{3}} d\tau ds \\ &\leq (CE_0)^{2n} + (C\delta)^{2n} \int_0^t (1+s)^l \|\nabla W(s)\|_{H^2}^{2n} ds \\ &\leq (CE_0)^{2n} + (C\delta)^{2n} \int_0^t (1+s)^l (\|\nabla W(s)\|_2^{2n} + \|\nabla^2 W(s)\|_{H^1}^{2n}) ds. \end{aligned} \quad (4.6)$$

Here we have used the fact

$$2n\left(\frac{3}{2p} - \frac{1}{4}\right) - l - 1 \geq 2n\left(\frac{3}{2p} - \frac{1}{4}\right) - 2n\left(\frac{3}{2p} - \frac{1}{4}\right) + 2 - 1 = 1.$$

Thus if $\delta > 0$ is sufficiently small such that $(C\delta)^{2n} \leq \frac{1}{2}$ in the final inequality of (4.6), then (4.6) implies (4.1). We finish the proof of Lemma 4.1. \square

Proof of Lemma 4.2. Multiplying (3.5) by $n(1+t)^l M(t)^{n-1}$ for $l = 0, 1, \dots, N$ and integrating it over $[0, t]$ give that

$$\begin{aligned}
& (1+t)^l M(t)^n + n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
& \leq M(0)^n + C\delta n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds + l \int_0^t (1+s)^{l-1} M(s)^n ds.
\end{aligned} \tag{4.7}$$

For the second term on the right hand side of (4.7), from the Young inequality, (3.6) and Lemma 4.1, it holds that for any $\eta > 0$,

$$\begin{aligned}
& \delta n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
& \leq \delta n \int_0^t (1+s)^l \left[\frac{n-1}{n} \eta M(s)^n + \frac{1}{n} \frac{1}{\eta^{n-1}} \|\nabla W(s)\|_2^{2n} \right] ds \\
& \leq \delta n C_2 \eta \int_0^t (1+s)^l M(s)^{n-1} (\|\nabla W(s)\|_2^2 + \|\nabla^2 W(s)\|_{H^1}^2) ds \\
& \quad + \delta \eta^{1-n} \int_0^t (1+s)^l \|\nabla W(s)\|_2^{2n} ds \\
& \leq \delta n C_2 \eta \int_0^t (1+s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
& \quad + \delta n C_2 \eta \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
& \quad + \delta \eta^{1-n} [(CE_0)^{2n} + (C\delta)^{2n} \int_0^t (1+s)^l \|\nabla^2 W(s)\|_{H^1}^{2n} ds] \\
& \leq \delta \eta^{1-n} (CE_0)^{2n} + \delta n C_2 \eta \int_0^t (1+s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
& \quad + \delta n C_2 \eta \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
& \quad + \delta \eta^{1-n} (C\delta)^{2n} C_2^{n-1} \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds.
\end{aligned} \tag{4.8}$$

Choosing $\eta = \frac{1}{2C_2}$ in (4.8), it holds that

$$\begin{aligned}
& \delta n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
& \leq 2\delta (2C_2)^{n-1} (CE_0)^{2n} + \delta n \left[1 + \frac{2}{n} (C\delta)^{2n} (2C_2^2)^{n-1} \right] \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
& \leq \delta (CE_0)^{2n} + \delta n \left[1 + (C\delta)^{2n} \right] \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds.
\end{aligned} \tag{4.9}$$

Thus if $\delta > 0$ is sufficiently small such that $C\delta \leq 1$ in (4.9), then $(C\delta)^{2n} \leq 1$ for any $n \geq 1$. And from (4.9), it is easy to show that

$$\begin{aligned} & \delta n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\ & \leq (CE_0)^{2n} + 2\delta n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds. \end{aligned} \quad (4.10)$$

Similarly, we can estimate for the third term on the right hand side of (4.7) as

$$\begin{aligned} & l \int_0^t (1+s)^{l-1} M(s)^n ds \\ & \leq (CE_0)^{2n} + \delta n (C\delta)^{2n-1} \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ & \quad + 2lC_2 \int_0^t (1+s)^{l-1} M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds, \end{aligned} \quad (4.11)$$

which together with (4.7) and (4.10) arrives at

$$\begin{aligned} & (1+t)^l M(t)^n + n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ & \leq M(0)^n + (CE_0)^{2n} + \delta n [C + (C\delta)^{2n-1}] \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ & \quad + 2lC_2 \int_0^t (1+s)^{l-1} M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds. \end{aligned} \quad (4.12)$$

Choose $\delta > 0$ sufficiently small such that for any $n \geq 1$, it follows that

$$\delta [C + (C\delta)^{2n-1}] \leq \frac{1}{2},$$

then it follows from (4.12) that

$$\begin{aligned} & (1+t)^l M(t)^n + n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ & \leq 2M(0)^n + (C_3 E_0)^{2n} + 4lC_2 \int_0^t (1+s)^{l-1} M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds, \end{aligned}$$

which gives (4.2) because $l \leq N \leq 2n(\frac{3}{2p} - \frac{1}{4})$. Thus we complete the proof of Lemma 4.2. \square

Proof of Theorem 1.2. Let $\delta > 0$ be small enough such that Lemma 4.2 holds for any $n \geq 2$. For any fixed integer $n \geq 2$, from Lemma 4.2, we get that the inequality (4.2) holds for any $l = 0, 1, \dots, N$. When $l = 1$, (4.2) reads

$$\begin{aligned} & (1+t)M(t)^n + n \int_0^t (1+s)M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ & \leq 2M(0)^n + (C_3 E_0)^{2n} + 8C_2 n \left(\frac{3}{2p} - \frac{1}{4}\right) \int_0^t M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds. \end{aligned} \quad (4.13)$$

It follows from (1.3) in Proposition 1.1 that

$$\begin{aligned} \int_0^t M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds &\leq [\sup_{s \geq 0} M(s)]^{n-1} \int_0^t \|\nabla^2 W(s)\|_{H^1}^2 ds \\ &\leq (C_2 C_0 \delta^2)^{n-1} C_0 \delta^2 \leq (C_2 C_0 \delta^2)^n. \end{aligned}$$

which together with (4.13) shows that

$$\begin{aligned} (1+t)M(t)^n + n \int_0^t (1+s)M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ \leq 2M(0)^n + (C_3 E_0)^{2n} + 8C_2 n \left(\frac{3}{2p} - \frac{1}{4}\right) (C_2 C_0 \delta^2)^n. \end{aligned} \quad (4.14)$$

For $1 \leq l \leq N$, by induction one can arrive at

$$\begin{aligned} (1+t)^l M(t)^n + n \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ \leq [2M(0)^n + (C_3 E_0)^{2n}] \sum_{i=1}^l [8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^i + n [8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^l (C_2 C_0 \delta^2)^n. \end{aligned} \quad (4.15)$$

In fact, suppose that (4.15) holds for $1 \leq l \leq N-1$. Then from (4.2), it holds that

$$\begin{aligned} (1+t)^{l+1} M(t)^n + n \int_0^t (1+s)^{l+1} M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ \leq 2M(0)^n + (C_3 E_0)^{2n} + 8C_2 n \left(\frac{3}{2p} - \frac{1}{4}\right) \int_0^t (1+s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\ \leq [2M(0)^n + (C_3 E_0)^{2n}] + 8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right) \{ [2M(0)^n + (C_3 E_0)^{2n}] \sum_{i=1}^l [8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^{i-1} \\ + n [8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^l (C_2 C_0 \delta^2)^n \} \\ \leq [2M(0)^n + (C_3 E_0)^{2n}] \sum_{i=1}^{l+1} [8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^{i-1} + n [8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^{l+1} (C_2 C_0 \delta^2)^n, \end{aligned} \quad (4.16)$$

which combining with (4.14) gives that (4.15) holds for any $1 \leq l \leq N$.

Specially,

$$(1+t)^N M(t)^n \leq [2M(0)^n + (C_3 E_0)^{2n}] \frac{[8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^N - 1}{8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right) - 1} + n [8C_2 \left(\frac{3}{2p} - \frac{1}{4}\right)]^N (C_2 C_0 \delta^2)^n.$$

Note that

$$2n \left(\frac{3}{2p} - \frac{1}{4}\right) - 3 \leq N = [2n \left(\frac{3}{2p} - \frac{1}{4}\right) - 2] \leq 2n \left(\frac{3}{2p} - \frac{1}{4}\right) - 1.$$

It is not difficult to prove that

$$(1+t)^{2n \left(\frac{3}{2p} - \frac{1}{4}\right) - 3} \leq C^{2n \left(\frac{3}{2p} - \frac{1}{4}\right)} [M(0)^n + E_0^{2n} + \delta^{2n}],$$

which gives that

$$M(t)^{\frac{1}{2}} \leq C [M(0)^n + E_0^{2n} + \delta^{2n}]^{\frac{1}{2n}} (1+t)^{-\left(\frac{3}{2p} - \frac{1}{4}\right) + \frac{3}{2n}}.$$

Since $M(0)$, E_0 and δ are independent of n , one gets

$$[M(0)^n + E_0^{2n} + \delta^{2n}]^{\frac{1}{2n}} \rightarrow \max\{\sqrt{M(0)}, E_0, \delta\}, \text{ as } n \rightarrow \infty.$$

The above relation implies that

$$M(t)^{\frac{1}{2}} \leq C \max\{\sqrt{M(0)}, E_0, \delta\} (1+t)^{-\left(\frac{3}{2p}-\frac{1}{4}\right)},$$

that is,

$$\|\nabla W(t)\|_{H^2} \leq C \max\{\sqrt{M(0)}, E_0, \delta\} (1+t)^{-\sigma(p,2;1)},$$

which together with Lemma 2.3 implies (1.7) and (1.8).

Now, estimate for (1.6). For this purpose, applying (1.8), Lemma 2.1 and Lemma 2.2 leads to

$$\begin{aligned} \|W(t)\|_2 &\leq CE_0(1+t)^{-\sigma(p,2;0)} + C \int_0^t (1+t-s)^{-\sigma(p,2;0)} (\|F(W)\|_p + \|F(W)\|_2) ds \\ &\leq CE_0(1+t)^{-\sigma(p,2;0)} + C\delta \int_0^t (1+t-s)^{-\sigma(p,2;0)} \|\nabla W(s)\|_{H^1} ds \\ &\leq CE_0(1+t)^{-\sigma(p,2;0)} + C\delta \int_0^t (1+t-s)^{-\sigma(p,2;0)} (1+s)^{-\sigma(p,2;1)} ds \\ &\leq C(1+t)^{-\sigma(p,2;0)}. \end{aligned}$$

By interpolation, we have that (1.6) holds for any $2 \leq q \leq 6$.

For (1.9), from (2.1), we get

$$\begin{aligned} \|\partial_t W(t)\|_2 &\leq \|\gamma \operatorname{div} v\|_2 + \|(F_1, F_2, F_3, F_4, F_5)\|_2 + \|\gamma \nabla a + \lambda \Delta v + \lambda \nabla \operatorname{div} v\|_2 \\ &\quad + \|\lambda \Delta h\|_2 + \|\lambda \Delta m\|_2 + \|\lambda \Delta \varepsilon\|_2 \\ &\leq C(\|\nabla(a, v, h, m, \varepsilon)\|_2 + \|\nabla^2(v, h, m, \varepsilon)\|_2) \\ &\leq CE_0(1+t)^{-\sigma(p,2;1)}. \end{aligned}$$

Thus, (1.9) is proved. The proof of Theorem 1.2 is complete. \square

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